International Journal of Theoretical Physics, Vol. 33, No. 10, 1994

Exact Traveling Wave Solutions to Nonlinear Diffusion and Wave Equations

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Received April 27, 1994

By the introduction of some ansatz equations, we have obtained several new classes of traveling (solitary) wave solutions to the nonlinear diffusion equation

$$f_1(u)u_t + f_2(u)u_x + f_3(u)u_{xx} + f_4(u)u_x^2 = f_5(u)$$

and the nonlinear wave equation

$$f_1(u)u_{tt} + f_2(u)u_t + f_3(u)u_{xx} + f_4(u)u_x + f_5(u)u_x^2 + \cdots = f_6(u)$$

Some applications of these solutions are discussed.

Nonlinear partial differential equations have been studied intensively (Drazin and Johnson, 1989; Sachdev, 1987; Newell and Moloney, 1992). Searching for traveling wave solutions to this class of equations is an important topic, as traveling wave solutions provide a description of the propagation and/or aggregation processes related to some physical systems. However, due to the mathematical complexity of nonlinear partial differential equations, few techniques have been introduced to obtain exact solutions (Drazin and Johnson, 1989; Sachdev, 1987; Newell and Moloney, 1992; Lu *et al.*, 1993; Ablowitz and Zeppetella, 1979; Murray, 1989; Hereman *et al.*, 1986; Hereman and Takaoka, 1990; Coffey, 1990, 1992; Wang, 1988; Wang *et al.*, 1990). The ansatz solution, reported in a recent letter by Lu *et al.* (1993), has shown that the nonlinear diffusion equation

$$f_1(u)u_t + f_2(u)u_x + f_3(u)u_{xx} + f_4(u)u_x^2 = f_5(u) \tag{1}$$

where f_i $(i = 1, 2, 3, 4, 5)$ are polynomials of u, $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$,

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0020-7748/94/1000-2057$07.00/0

$u_{xx} = \partial^2 u/\partial x^2$, has a general traveling wave solution

$$u(y) = u(x - ct) = \left[-\frac{a}{2b} \tanh\left(\frac{n-1}{2} a(x - ct - c_0)\right) - \frac{a}{2b} \right]^{1/(n-1)} \quad (2)$$

for the condition

$$[f_2(u) - cf_1(u) + (a + bnu^{n-1})f_3(u) + (au + bu^n)f_4(u)](au + bu^n) \equiv f_5(u) \quad (3)$$

In the above c is the speed of the propagating waves, c_0 is a constant usually determined by the initial conditions, and a, b, and n are real constants which satisfy the conditions $n \neq 1$ and $ab < 0$.

In this paper we introduce some new ansatz equations which have not been considered in Lu *et al.* (1993) and obtain several new classes of traveling wave solutions for conditions which differ from equation (3). We also discuss the traveling wave solutions to some nonlinear wave equations. In the following sections we present the results of our present investigations. We begin by extending the scope of application of the traveling wave solution (2) to some nonlinear wave equations. We then introduce four new ansatz equations to solve the nonlinear diffusion equations and nonlinear wave equations. The conditions related to these ansatz solutions are also given. In addition we present some examples following each of the ansatz solutions of some possible applications to the description of physical systems.

1. Let us consider the dissipative nonlinear wave equation

$$f_1(u)u_{tt} + f_2(u)U_t + f_3(u)u_{xx} + f_4(u)u_x + f_5(u)u_x^2 + \cdots = f_6(u) \quad (4)$$

where $u_{tt} = \partial^2 u/\partial t^2$, and the coefficient functions $f_1, f_2, f_3, f_4, f_5, f_6$, etc., are algebraic functions of u. It can be shown that equation (2) is a traveling wave solution to equation (4) for the condition [similar to equation (3)]

$$\{f_4(u) - cf_2(u) + (a + bnu^{n-1})[c^2 f_1(u) + f_3(u)] + (au + bu^n)f_5(u) + \cdots\}(au + bu^n) \equiv f_6(u) \quad (5)$$

In fact, using the transform $\xi = x - ct$, we find that equation (4) has the form

$$c^2 f_1(u)u'' + [f_4(u) - cf_2(u)]u' + f_3(u)u'' + f_5(u)(u')^2 + \cdots = f_6(u) \quad (6)$$

where $u' = du/dy$ and $u'' = d^2u/dy^2$. The ansatz introduced by Lu *et al.* (1993) is

$$u' = au + bu^n \quad (7)$$

which is the Bernoulli equation with constant coefficients. Thus, one has

$$u'' = a^2 u + ab(n + 1)u^n + b^2 n u^{2n-1} = (a + nbu^{n-1})u' \quad (8)$$

Equation (5) is obtained by substituting these results into equation (6). In order to illustrate the utilization of the technique to obtain traveling wave solutions for a particular problem, we present the following example.

Example 1. Let us consider the dissipative nonlinear Klein–Gordon equation

$$u_{xx} - \gamma u_t - u_{tt} = \alpha u - \beta u^3 + \delta u^5$$

where α, β, γ, and δ are real numbers which satisfy $\beta \geq 0$ and $\gamma, \delta > 0$. Without the dissipation term, this equation is the higher-order approximation comparing with the standard ϕ^4-model widely used in the field theory (Lee, 1988).

First, we consider the nondissipative case, i.e., $\gamma = 0$. When $3\beta^2 = 16\alpha\delta$, we obtain the traveling wave solution as

$$u(x - ct) = \left\{\frac{1}{2}\left(\frac{3\alpha}{\delta}\right)^{1/2} \tanh\left[\pm\left(\frac{\alpha}{1 - c^2}\right)^{1/2}(x - ct - c_0)\right] + \frac{1}{2}\left(\frac{3\alpha}{\delta}\right)^{1/2}\right\}^{1/2}$$

When the dissipation term is introduced, we obtain the traveling solution as

$$u(x - ct) = \frac{1}{2}\left[\frac{3}{\delta(1 - c^2)}\right]^{1/4} \{[\gamma^2 c^2 + 4\alpha(1 - c^2)]^{1/2} \mp \gamma c\}^{1/2}$$
$$\times \left\{1 \pm \tanh\left[\frac{[\gamma^2 c^2 + 4\alpha(1 - c^2)]^{1/2} \mp \gamma c}{2(1 - c^2)}(x - ct - c_0)\right]\right\}^{1/2}$$

where the speed of the wave c is given by the relationship

$$c^2 = \frac{256\alpha^2\delta^2 - 96\alpha\beta^2\delta + 9\beta^4 - 48\alpha\gamma^2\delta^2 + 15\beta^2\gamma^2\delta \pm 12\beta\gamma^2\delta(\delta^2 - 4\alpha\delta)^{1/2}}{256\alpha^2\delta^2 - 96\alpha\beta^2\delta + 9\beta^4 - 96\alpha\gamma^2\delta^2 + 30\beta^2\gamma^2\delta + 9\gamma^4\delta^2}$$

Comparing the solutions to the nondissipative and dissipative equations, it is easy to see that the dissipation term changes the system dramatically, although both have the kink-type traveling wave representations. Furthermore, the speed of the waves c is an arbitrary constant for the nondissipative system, while it is restricted by the coefficients for the dissipative system.

Example 2. Let us consider the elastic-medium wave equation (Drazin and Johnson, 1989)

$$u_{tt} - u_{xx} - u_x u_{xx} - u_{xxxx} = 0$$

It is straightforward to obtain the traveling wave solution as

$$u(x - ct) = 6(c^2 - 1)^{1/2} \tanh\left[\frac{(c^2 - 1)^{1/2}}{2}(x - ct - c_0)\right] \pm 6(c^2 - 1)^{1/2}$$

2. We can show that equations (1) and (4) have a different class of traveling wave solutions of the form

$$u(y) = u(x - ct) = \frac{\sqrt{\Delta}}{2a_2} \tan\left[\frac{\sqrt{\Delta}}{2}(x - ct - c_0)\right] - \frac{a_1}{2a_2} \qquad (9)$$

where a_0, a_1, and a_2 are real numbers satisfying the relationship $\Delta = 4a_0a_2 - a_1^2 > 0$ under the conditions

$$\begin{aligned}&[f_2(u) - cf_1(u) + (a_1 + 2a_2u)f_3(u) + (a_0 + a_1u + a_2u^2)f_4(u)]\\ &\quad \times (a_0 + a_1u + a_2u^2) \equiv f_5(u)\end{aligned} \qquad (10)$$

for equation (1) and

$$\begin{aligned}&\{f_4(u) - cf_2(u) + (a_1 + 2a_2u)[c^2f_1(u) + f_3(u)]\\ &\quad + (a_0 + a_1u + a_2u^2)f_5(u) + \cdots\}(a_0 + a_1u + a_2u^2) \equiv f_6(u)\end{aligned} \qquad (11)$$

for equation (4). Similarly, if we allow $y = x - ct$, then equation (1) has the form

$$-cf_1(u)u' + f_2(u)u' + f_3(u)u'' + f_4(u)(u')^2 = f_5(u) \qquad (12)$$

We introduce the ansatz

$$u' = a_0 + a_1u + a_2u^2 \qquad (13)$$

which is the Riccati equation with constant coefficients. Thus, we have

$$\begin{aligned}u'' &= a_0a_1 + (2a_0a_2 + a_1^2)u + 3a_1a_2u^2 + 2a_2^2u^3\\ &= (a_1 + 2a_2u)u'\end{aligned} \qquad (14)$$

Equations (10) and (11) can be obtained by substituting equations (13) and (14) into equations (12) and (6). In the case of $\Delta = 4a_0a_2 - a_1^2 < 0$, the Riccati equation (13) can be transformed to the Bernoulli equation, and its solution can be expressed as

$$u(y) = u(x - ct) = -\frac{\sqrt{-\Delta}}{2a_2} \tanh\left[\frac{\sqrt{-\Delta}}{2}(x - ct - c_0)\right] - \frac{a_1}{2a_2} \qquad (15)$$

It is important to point out that the solution for $\Delta = 4a_0a_2 - a_1^2 < 0$ is not a trivial case of Case 1 above when $n = 2$, which corresponds to $a_0 = 0$ in the ansatz given by equation (13). In the following two examples this point will be demonstrated.

Example 3. Let us consider the dissipative ϕ^4-model equation (Lee, 1988)

$$u_{xx} - \gamma u_t - u_{tt} = \alpha u + \beta u^3$$

where α, β, and γ are real numbers which satisfy $\alpha < 0$ and $\beta, \gamma > 0$.

When $\gamma = 0$, it is easy to see that the ansatz of Case 1 is not applicable to the problem. Using Case 2, some algebra calculations yield the traveling wave solutions to the nondissipative ϕ^4-model equation of the form

$$u(x-ct) = \left(\frac{-\alpha}{\beta}\right)^{1/2} \tanh\left[\left(\frac{-\alpha}{2(1-c^2)}\right)^{1/2}(x-ct-c_0)\right] \quad \text{for} \quad \alpha < 0$$

$$u(x-ct) = \left(\frac{\alpha}{\beta}\right)^{1/2} \tan\left[\left(\frac{\alpha}{2(1-c^2)}\right)^{1/2}(x-ct-c_0)\right] \quad \text{for} \quad \alpha > 0$$

When the dissipative term is introduced, the traveling solutions have been changed dramatically as

$$u(x-ct) = \frac{1}{2}\left(\frac{-\alpha}{\beta}\right)^{1/2} \tanh\left\{\frac{[-\alpha(2\gamma^2-9\alpha)]^{1/2}}{4\gamma} \times \left[x \mp \left(\frac{-9\alpha}{2\gamma^2-9\alpha}\right)^{1/2} t - c_0\right]\right\} + \frac{1}{2}\left(\frac{-\alpha}{\beta}\right)^{1/2} \quad \text{for} \quad \alpha < 0$$

$$u(x-ct) = \frac{1}{2}\left(\frac{\alpha}{\beta}\right)^{1/2} \tan\left\{\frac{[\alpha(2\gamma^2-9\alpha)]^{1/2}}{4\gamma} \times \left[x \mp \left(\frac{9\alpha}{2\gamma^2-9\alpha}\right)^{1/2} t - c_0\right]\right\} + \frac{1}{2}\left(\frac{\alpha}{\beta}\right)^{1/2} \quad \text{for} \quad \alpha > 0$$

Example 4. Let us consider a generalized Fisher equation of the form

$$u_t - \alpha u_{xx} = \beta u - \gamma u^2 - \delta u^3$$

where α, β, γ, and δ are positive real numbers. If we consider Case 1, we obtain the solutions

$$u(x-ct) = \frac{\gamma \pm (4\beta\delta+\gamma^2)^{1/2}}{4\delta} \tanh\left[\frac{\gamma \pm (4\beta\delta+\gamma^2)^{1/2}}{(8\alpha\delta)^{1/2}} x - \left(\frac{\beta}{2} + \frac{\gamma^2 + 2\beta\delta \pm \gamma(4\beta\delta+\gamma^2)^{1/2}}{2\delta}\right) t - c_0\right] - \frac{\gamma \pm (4\beta\delta+\gamma^2)^{1/2}}{4\delta}$$

and

$$u(x-ct) = -\frac{\gamma \pm (4\beta\delta+\gamma^2)^{1/2}}{4\delta} \tanh\left[\frac{\gamma \pm (4\beta\delta+\gamma^2)^{1/2}}{(8\alpha\delta)^{1/2}} x + \left(\frac{\beta}{2} + \frac{\gamma^2 + 2\beta\delta \pm \gamma(4\beta\delta+\gamma^2)^{1/2}}{2\delta}\right) t - c_0\right] - \frac{\gamma \pm (4\beta\delta+\gamma^2)^{1/2}}{4\delta}$$

However, using the ansatz given in Case 2, we obtain a new solution:

$$u(x-ct) = \pm\frac{(4\beta\delta+\gamma^2)^{1/2}}{2\delta} \times \tanh\left\{\left(\frac{4\beta\delta+\gamma^2}{8\alpha\delta}\right)^{1/2}\left[x \pm \gamma\left(\frac{\alpha}{2\delta}\right)^{1/2} t - c_0\right]\right\} - \frac{\gamma}{2\delta}$$

3. This technique may be used to solve some nonlinear diffusion equations and wave equations. We consider equations (1) and (4), with coefficient functions $f_1, f_2, f_3, f_4, f_5, f_6$, etc., which are algebraic functions of u satisfying the relations

$$\left[f_2(u) - cf_1(u) - \frac{u}{(a^2-u^2)^{1/2}} f_3(u) + (a^2-u^2)^{1/2} f_4(u)\right](a^2-u^2)^{1/2} \equiv f_5(u) \tag{16}$$

for equation (1) and

$$\left\{f_4(u) - cf_2(u) - \frac{u}{(a^2-u^2)^{1/2}}[c^2 f_1(u) + f_3(u)] + (a^2-u^2)^{1/2} f_5(u) + \cdots\right\}(a^2-u^2)^{1/2} \equiv f_6(u) \tag{17}$$

for equation (4). Introducing the ansatz

$$u' = (a^2-u^2)^{1/2} \tag{18}$$

we obtain the solution as

$$u(y) = u(x-ct) = a\sin(x-ct-c_0) \tag{19}$$

This is a harmonic wave solution which only differs from the linear wave equation in the amplitude, which is determined by the nonlinear equation in the present case, while it is controlled by the initial conditions for the linear wave equation.

Example 5. Let us consider the following dissipative nonlinear wave equation:

$$u_{tt} - u_{xx} + (\alpha^2-u^2)^{1/2} u_t + u_x^2 = 0$$

Some straightforward algebra yields

$$-c(\alpha^2-u^2)^{1/2} - \frac{u(c^2-1)}{(a^2-u^2)^{1/2}} + (a^2-u^2)^{1/2} \equiv 0$$

This equality requires $a = \alpha$ and $c = 1$. The harmonic wave solution to this equation is given by

$$u(x-ct) = \alpha\sin(x-t-c_0)$$

4. This technique may be extended to more complex cases of the nonlinear diffusion equation and nonlinear wave equation. We consider equations (1) and (4) with coefficient functions $f_1, f_2, f_3, f_4, f_5, f_6$, etc., which are triangle functions of u satisfying the relations

$$[f_2(u) - cf_1(u) + ab\cos(bu) f_3(u) + a\sin(bu) f_4(u)]a\sin(bu) \equiv f_5(u) \tag{20}$$

for equation (1) and

$$\{f_4(u) - cf_2(u) + ab\cos(bu)[c^2 f_1(u) + f_3(u)] \\ + a\sin(bu)f_5(u) + \cdots\}a\sin(bu) \equiv f_6(u) \tag{21}$$

for equation (4). Introducing the ansatz

$$u' = a\sin(bu) \tag{22}$$

we obtain the solitary wave solution

$$u(y) = u(x - ct) = 2b^{-1}\arctan\{\exp[ab(x - ct - c_0)]\} \tag{23}$$

Two practical examples of this case are as follows.

Example 6. Consider the sine-Gordon equation

$$u_{xt} = \alpha\sin(u)$$

where α is a real number. This equation has been used widely in the physical sciences (Grønbech-Jensen *et al.*, 1993; Sørensen *et al.*, 1993; Wang and Yao, 1993). Using the ansatz given by (22), we can easily obtain the solution as

$$u(x - ct) = 4\arctan\left\{\exp\left[\pm\left(\frac{-\alpha}{c}\right)^{1/2}(x - ct) - c_0\right]\right\}$$

Example 7. Consider a dissipative sine-Gordon equation

$$u_{xx} - \gamma u_t - u_{tt} = \alpha_1\sin(\beta u) + \alpha_2\sin(2\beta u)$$

where α_1, α_2, β, and γ are positive real constants. This equation may be regarded as a phenomenological representation of the dissipation and a higher-order approximation compared with the original sine-Gordon equation. It is easy to obtain the solitary wave solutions as

$$u(x - ct) = \frac{2}{\beta}\arctan\left(\exp\left\{\pm\left[\frac{\beta(2\alpha_2\gamma^2 + \alpha_1^2\beta)}{\gamma^2}\right]^{1/2} x - \frac{\alpha_1\beta}{\gamma}t - c_0\right\}\right)$$

When $\alpha_2 = 0$, this solution reduces to

$$u(x - ct) = \frac{2}{\beta}\arctan\left\{\exp\left[\pm\frac{\alpha_1\beta}{\gamma}(x \mp t - c_0)\right]\right\}$$

5. We now give another example for the complex cases of the nonlinear diffusion equation and the nonlinear wave equation. We consider equations (1) and (4), with coefficient functions $f_1, f_2, f_3, f_4, f_5, f_6$, etc., which are also triangle functions of u satisfying the relation

$$[f_2(u) - cf_1(u) + ab\sin(2bu)f_3(u) + a\sin^2(bu)f_4(u)]a\sin^2(bu) \equiv f_5(u) \tag{24}$$

for equation (1) and

$$\{f_4(u) - cf_2(u) + ab\sin(2bu)[c^2 f_1(u) + f_3(u)] + a\sin^2(bu)f_5(u) + \cdots\}a\sin^2(bu) \equiv f_6(u) \tag{25}$$

for equation (4). By introducing the ansatz

$$u' = a\sin^2(bu) = \frac{a}{2}[1 - \cos(2bu)] \tag{26}$$

we obtain the solitary wave solution as

$$u(y) = u(x - ct) = -b^{-1}\operatorname{arccot}[ab(x - ct - c_0)] \tag{27}$$

Example 8. Similarly, we consider a sine-Gordon-like equation

$$u_{xx} - \gamma u_t - u_{tt} = \alpha_1 \sin^2(\beta u) + \alpha_2 \sin^3(\beta u)\cos(\beta u)$$

where α_1, α_2, and β are real constants. It is easy to obtain the solitary wave solution as

$$u(x - ct) = -\beta^{-1}\operatorname{arccot}\left\{\pm\left[\frac{(2\alpha_1^2\beta + \alpha_2\gamma^2)\beta}{2\gamma^2}\right]^{1/2} x - \frac{\alpha_1}{\gamma}t - c_0\right\}$$

When $\alpha_2 = 0$, this reduces to

$$u(x - ct) = -\beta^{-1}\operatorname{arccot}\left[\pm\frac{\alpha_1\beta}{\gamma}\left(x \mp \frac{1}{\beta}t - c_0\right)\right]$$

In this case, the speed of the propagating wave $c = 1/\beta$ is independent of the damping γ and coupling factor α_1.

It is worthwhile to point out that, as illustrated in Examples 1, 3, 7, and 8, the dissipation term can change the traveling wave solutions dramatically due to the high nonlinearity of the equations.

It has been generally shown that the $(1 + 1)$-dimensional traveling wave solutions could be generalized to some higher-dimension equations (Drazin and Johnson, 1989). For the same reason, the ansatz solutions given in this paper can easily be generalized to higher-dimension equations as well.

In summary, we have extended the scope of the application of the ansatz solution given by Lu *et al.* By introducing four new ansatz equations, we have obtained four new classes of traveling (and/or solitary) wave solutions to some nonlinear diffusion equations and nonlinear wave equations. We have also presented typical examples to illustrate the application of these ansatz solutions to some particular problems.

ACKNOWLEDGMENTS

This research was supported in part by the Killam Foundation, the Natural Science and Engineering Research Council (NSERC) of Canada, and the Faculty of Graduate Studies, Dalhousie University.

REFERENCES

Ablowitz, M. J., and Zeppetella, A. (1979). *Bulletin of Mathematical Biology*, **41**, 835.

Coffey, M. W. (1990). *SIAM Journal of Applied Mathematics*, **50**, 1580.

Coffey, M. W. (1992). *SIAM Journal of Applied Mathematics*, **52**, 929.

Drazin, P. G., and Johnson, R. S. (1989). *Solitons: An Introduction*, Cambridge University Press, Cambridge.

Grønbech-Jensen, N., Olsen, O. H., and Samuelsen, M. R. (1993). *Physics Letters A*, **179**, 27.

Hereman, W., and Takaoka, M. (1990). *Journal of Physics A*, **23**, 4805.

Hereman, W., Banerjee, P. P., Korpel, A., Assanto, G., van Immerzeele, A., and Meerpoel, A. (1986). *Journal of Physics A*, **19**, 607.

Lee, T. D. (1988). *Particle Physics and Introduction to Field Theory*, Harwood Academic, London.

Lu, B. Q., Xiu, B. Z., Pang, Z. L., and Jiang, X. F. (1993). *Physics Letters A*, **175**, 113.

Murray, J. D. (1989). *Mathematical Biology*, Springer, Berlin.

Newell, A. C., and Moloney, J. V. (1992). *Nonlinear Optics*, Addison-Wesley, Redwood, California.

Sachdev, P. L. (1987). *Nonlinear Diffusive Waves*, Cambridge University Press, Cambridge.

Sørensen, M. P., Malomed, B. A., Ustinov, A. V., and Pedersen, N. F. (1993). *Physica D*, **68**, 38.

Wang, X. Y. (1988). *Physics Letters A*, **131**, 277.

Wang, Q. H., and Yao, X. X. (1993). *Physical Review B*, **48**, 6334.

Wang, X. Y., Chu, Z. S., and Lu, Y. K. (1990). *Journal of Physics A*, **23**, 271.